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Remarks on a theorem of Swarup on ends of pairs of groups

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Abstract

We give an algebraic proof of a recent theorem of Swarup, which states that if H is a subgroup of infinite index in a finitely generated group G and if $e(G, N) = 1$ for all subgroups $N \lhd H$ with $H/N \cong \mathbb{Z}$, then $e(G, H) = 1 + \text{rank } H^1(G, \mathbb{Z}[H\backslash G])$. We also consider some generalizations of this theorem.

1. Introduction

Throughout, G denotes a group and H denotes a subgroup of infinite index. Our main interest is the study of the number of ends $e(G, H)$ of the pair (G, H).

We briefly recall some definitions. A subset E of G is called H-finite if it is contained in a finite union of right cosets of H, that is $E \subseteq HF$ for some finite $F \subseteq G$. The symmetric difference of two subsets E and E' is denoted by $E + E'$. We say that E is H-almost invariant if and only if $E + Ex$ is H-finite for all $x \in G$. The ends of the pair (G, H) correspond, in effect, to the H-almost invariant subsets E which satisfy $E = HE$ and which are not H-finite. We refer the reader to [1, 2] for a more detailed account.

1.1. *Let E be an H-almost invariant subset of G and let $N = \{h \in H \mid hE = E\}$. Suppose that the following conditions hold.*

(1) *N is a normal subgroup of H;*
(2) *$hE \cap E = \emptyset$, for all $h \in H\backslash N$; and*
(3) *$HE = G$.*

Then $e(G, N) \geq |H : N|$.

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Proof. First note that for any $x, y \in G$, $Ex \cap Hy$ is contained in single coset of N. To see this, suppose given $z, z' \in Ex \cap Hy$. Then $z'z^{-1}$ belongs to H and also $z'z^{-1}E \cap E$ contains $z'x^{-1}$ and so is non-empty. Hence $z'z^{-1}$ belongs to N.

Now for any $x \in G$, $E + Ex$ is H-finite, and so it follows from the above that it is actually N-finite. Hence E is N-almost invariant. Moreover, E is not N-finite, for otherwise $G = HE$ would be H-finite, contradicting the assumption that H has infinite index in G. Since N is normal in H, each of the subsets hE, for $h \in H$, corresponds to an end of the pair (G, N) and since these sets are in natural bijective correspondence with H/N we have $e(G, N) \geq |H:N|$. □

2. From derivations to almost invariant sets

For a group A we write $C(A)$ for the set of functions from G to A which are constant on left cosets gH of H, and we write $I(A)$ for the subset of $C(A)$ consisting of functions which are supported on finitely many left cosets. $C(A)$ and $I(A)$ are themselves groups under pointwise multiplication, and admit an action of G given by $\phi^g(g') = \phi(gg')$ for $\phi \in C(A)$ and $g, g' \in G$. A derivation $\delta: G \to I(A)$ is a function satisfying $\delta(gg') = (\delta g)^{g'}(\delta g')$. For such a derivation, we write δ^* for the function from G to A defined by

$$\delta^*(g) := \delta g(1).$$

In general, δ^* is not a homomorphism, but it does at least satisfy the following.

2.1. *For all $g \in G$ and $h \in H$,*

$$\delta^*(gh) = \delta^*(g)\delta^*(h).$$

For each subset Y of A, let E_Y be the subset of G defined by

$$E_Y := \{g \in G \mid \delta^*(g^{-1}) \in Y\}.$$

2.2. *For all $Y \subseteq A$, E_Y is H-almost invariant.*

Proof. Fix Y. For convenience, we write $E := E_Y$. We must show that for all $x \in G$, the symmetric difference $E + Ex$ is H-finite. For $g \in G$, g belongs to $E + Ex$ if and only if exactly one of g and gx^{-1} belongs to E, or equivalently, if and only if exactly one of $\delta^*(g^{-1})$ and $\delta^*(xg^{-1})$ belongs to Y. Since $\delta^*(xg^{-1}) = \delta x(g^{-1})\delta^*(g^{-1})$ it follows that $\delta x(g^{-1})$ is non-trivial for every $g \in E + Ex$. Hence $E + Ex \subseteq (\mathrm{supp}(\delta x))^{-1}$ and this is H-finite as required. □

2.3. *For all $h \in H$ and $Y \subseteq A$,*

$$hE_Y = E_{Y(\delta^* h)^{-1}}.$$

Proof. Let g be any element of G. Then g belongs to hE if and only if $\delta^*(g^{-1}h) \in Y$. Since $\delta^*(g^{-1}h) = \delta^*(g^{-1})\delta^*(h)$ by (2.1), it follows that

$$hE = \{g \in G \mid \delta^*(g^{-1}) \in Y(\delta^* h)^{-1}\} = E_{Y(\delta^* h)^{-1}}$$

as required. □

2.4. *For subsets Y and Y' of A, $E_Y \cap E_{Y'} = E_{Y \cap Y'}$.*

2.5. *Let N denote the kernel of $\delta^*|_H$. Then $e(G, N) \geq |H:N|$.*

Proof. Let B denote the image of $\delta^*|_H$ and let Y be a transversal to B in A, so that A is the disjoint union of the cosets yB, $y \in Y$. If $h \in H$ and $\delta^* h = b$ then (2.3) shows that $hE_Y = E_{Yb^{-1}}$. If b is non-trivial then $Y \cap Yb^{-1} = \emptyset$ and it follows from (2.4) that $E_Y \cap E_{Yb^{-1}} = \emptyset$. It also follows that $N = \{h \in H \mid hE_Y = E_Y\}$. Thus, the result follows by applying (1.1) with $E := E_Y$. □

3. Swarup's theorem and generalizations

Throughout this section we assume that G is finitely generated, and we let A denote an abelian group, written additively. In this case, $I(A)$ and $C(A)$ can be identified with the induced and coinduced modules $\mathrm{Ind}_H^G(A)$ and $\mathrm{Coind}_H^G(A)$. The inclusion of $I(A)$ into $C(A)$ induces a map

$$\rho : H^1(G, I(A)) \to H^1(G, C(A)).$$

We refer the reader to [2, 3] for details concerning the next lemma. This result depends on G being finitely generated and on H having infinite index.

3.1. *Let A be either a prime field or a subring of $\mathbb{Q}$. Then $e(G, H) = 1 + \operatorname{rank} \operatorname{Ker} \rho$*

Using the Shapiro–Eckmann lemma, $H^1(G, C(A))$ can be identified with $\mathrm{Hom}(H, A)$, and through this identification the map ρ is given by

$$\rho[\delta] = \delta^*|_H,$$

where $\delta : G \to I(A)$ is a derivation representing a cohomology class $[\delta] \in H^1(G, I(A))$. Swarup's theorem [3] now follows at once by combining the above with (2.5).

3.2. (Swarup's theorem). *Suppose that $e(G, N) = 1$ for all $N \lhd H$ with $H/N \cong \mathbb{Z}$. Then $e(G, H) = 1 + \operatorname{rank} H^1(G, I(\mathbb{Z}))$.*

Proof. In view of (3.1), it suffices to show that $\rho : H^1(G, I(\mathbb{Z})) \to \mathrm{Hom}(H, \mathbb{Z})$ is trivial. Let $\delta : G \to I(\mathbb{Z})$ be a derivation. Then (2.5) shows that $e(G, N) = |H:N|$, where $N = \operatorname{Ker} \delta^*|_H$. If $\delta^*|_H$ is non-trivial then H/N is isomorphic to $\mathbb{Z}$ and $e(G, N) = \infty$, contrary to hypothesis. Therefore $\rho[\delta] = \delta^*|_H$ is trivial for all δ. □

We also have the following generalization.

3.3. *Suppose that the image of* $\rho: H^1(G, I(\mathbb{Z})) \to \mathrm{Hom}(H, \mathbb{Z})$ *has rank* $n \geq 1$. *Then there is a normal subgroup* N *of* H *such that* $H/N \cong \mathbb{Z}^n$ *and* $e(G, N) = \infty$.

Proof. Choose derivations $\delta_1, \ldots, \delta_n : G \to I(\mathbb{Z})$ so that $\delta_1^*|_H, \ldots, \delta_n^*|_H$ are linearly independent elements of $\mathrm{Hom}(H, \mathbb{Z})$. Now let A be $\mathbb{Z}^n$ and let δ be the derivation from G to $I(A)$ defined by

$$\delta g(g') = (\delta_1 g(g'), \ldots, \delta_n g(g')).$$

Let N be the kernel of $\delta^*|_H : H \to A$. Then $H/N \cong \mathbb{Z}^n$ and (2.5) shows that $e(G, N) = \infty$. $\square$

Working over finite fields one can obtained similar results by the same arguments.

3.4. *Let* $\mathbb{F}_p$ *denote the Galois field of p elements where p is a prime. If* $e(G, N) = 1$ *for all* $N \lhd H$ *with* $|H:N| = p$, *then* $e(G, H) = 1 + \mathrm{rank}\, H^1(G, I(\mathbb{F}_p))$.

3.5. *Suppose that the image of* $\rho: H^1(G, I(\mathbb{F}_p)) \to \mathrm{Hom}(H, \mathbb{F}_p)$ *has rank* $n \geq 1$. *Then there is a normal subgroup* N *of* H *such that* H/N *is an elementary abelian p-group of rank n and* $e(G, N) \geq p^n$.

4. Concluding remark

In [1] a new end invariant $\tilde{e}(G, H)$ was introduced and it is easy to see that

$$e(G, H) \geq 2 \Rightarrow H^1(G, I(\mathbb{F}_2)) \neq 0 \Rightarrow \tilde{e}(G, H) \geq 2.$$

Swarup's theorem, and the results of this paper provide a sharpening of the second implication here, by giving more precise information about how non-zero elements of $H^1(G, I(\mathbb{F}_2))$ contribute to $\tilde{e}(G, H)$.

References

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